

Determining All Universal Tilers

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Abstract

A universal tiler is a convex polyhedron whose every cross-section tiles the plane. In this paper, we introduce a certain slight-rotating operation for cross-sections of pentahedra. Based on a selected initial cross-section and by applying the slight-rotating operation suitably, we prove that a convex polyhedron is a universal tiler if and only if it is a tetrahedron or a triangular prism.

Keywords: cross-section, the slight-rotating operation, universal tiler

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1 Introduction

A tiler is a polygon that can cover the plane by congruent repetitions without gaps or overlaps. The problem of determining all tilers, called alternatively the problem of tessellation or plane tiling, is one of the most famous problems in discrete mathematics, and is still open to the best of our knowledge. For a whole theory of tessellation, see Grunbaum and Shephard's book [2].

Considering a variant of the problem of plane tiling, Akiyama [1] found all convex polyhedra whose every development is a tiler. The key idea in his proof is to find whether the facets of a polyhedron tile the plane in certain stamping manner. Noticing that facets are special cross-sections, we study another variant of the problem of plane tiling that what kind of polyhedra are so well-performed that every its cross-section is a tiler.

Let \mathcal{P} be a convex polyhedron, and π a plane. Denote the intersection of π and \mathcal{P} by $C(\pi)$. We say that π intersects \mathcal{P} non-trivially if $C(\pi)$ is a non-degenerated polygon, that is, $C(\pi)$ has at least 3 edges. We call $C(\pi)$ a *cross-section* if π crosses \mathcal{P} nontrivially. The polyhedron \mathcal{P} is said to be a *universal tiler* if every cross-section of \mathcal{P} is a tiler. In this paper, we will determine all universal tilers.

A triangular prism is a pentahedron with parallel facets. With the aid of Euler's formula, and Reinhardt's theorem [4] for the results on tilers with n ($n \neq 5$) edges, the author [6] managed to obtain the following necessary condition for the number of faces of a universal tiler by suitably choosing cross-sections of a given polyhedron.

Theorem 1.1 *A convex polyhedron is a universal tiler only if it is a tetrahedron or a pentahedron. Moreover, every tetrahedron and every triangular prism is a universal tiler.*

In light of the above theorem, the problem of determining all universal tilers turns out to be the one of finding the list of pentahedron universal tilers. One of the difficulties in determining whether a pentahedron is a universal tiler is the fact that no one knows the list of pentagonal tilers, although there are 14 classes of pentagonal tilers are found, see Hirschhorn and Hunt [3], and Sugimoto and Ogawa [5] for instance.

The key idea used in solving the universal tiler problem consists of two parts. One is to select an initial cross-section from a given pentahedron subject to some technical conditions. It is an extension of the method adopted in [6]. The other is to suitably apply a certain *slight-rotating operation* based on the initial cross-section. By suitably applying the operation at most three times, and considering the local situations of the tessellations, we can prove that only triangular prisms are pentahedron universal tilers. The whole proof has nothing to do with the knowledge of the complete list of pentagonal tilers.

This paper is organized as follows. In the next section, we give necessary notion and notation on tessellations of the plane by a single polygon. Section 3 is devoted to select the initial cross-section subject to some technical conditions and to introduce the slight-rotating operation. In the last section, by applying the slight-rotating operation we prove that one may invariantly obtain a non-tiler cross-section of a pentahedron \mathcal{P} unless \mathcal{P} is a triangular prism.

2 Preliminary

In this section, we introduce some necessary notion and notation. Suppose that

$$T = V^1 V^2 \dots V^5$$

is a pentagonal tiler. Let \mathcal{T} be a tessellation of the plane by copies of T . Denote the copies used in \mathcal{T} by

$$\{T_i = V_i^1 V_i^2 \dots V_i^5 : i \in \Lambda\},$$

where Λ is a set. Then every T_i has the same shape as T .

Let $i \in \Lambda$ and $\varepsilon > 0$. Since T is a tiler, the ε -neighborhood of the point V_i^j in \mathcal{T} must be covered without gaps or overlaps. It follows that either there is a sequence

$$V_i^j, V_{i_1}^{j_1}, V_{i_2}^{j_2}, \dots, V_{i_k}^{j_k}$$

($k \geq 2$) of angles arranged counter-clockwise which fulfilled the whole 2π -area around the point V_i^j (see the left part of Figure 1), or there is a sequence

$$V_{i_1}^{j_1}, \dots, V_{i_k}^{j_k}, V_i^j, V_{s_1}^{t_1}, \dots, V_{s_h}^{t_h},$$

($k + h \geq 1$) of angles arranged counter-clockwise which fulfilled a π -area around V_i^j (see the right part of Figure 1). In the former case, we denote the local tessellation around the point V_i^j by

$$S(V_i^j) = [V_i^j, V_{i_1}^{j_1}, V_{i_2}^{j_2}, \dots, V_{i_k}^{j_k}]. \quad (2.1)$$

In the latter case, we denote

$$S(V_i^j) = [V_i^j, V_{s_1}^{t_1}, \dots, V_{s_h}^{t_h}, \pi, V_{i_1}^{j_1}, \dots, V_{i_k}^{j_k}]. \quad (2.2)$$

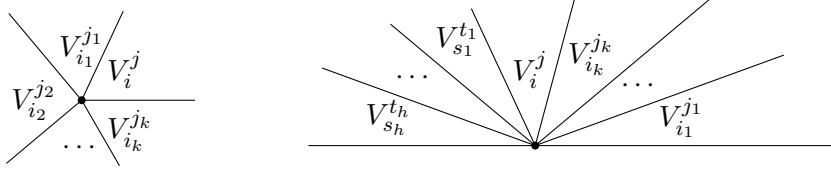


Figure 1. The local tessellation around the point V_i^j .

Suppose that there is a sequence

$$V_{i_1}^{j_1} V_{i_1}^{k_1}, \quad V_{i_2}^{j_2} V_{i_2}^{k_2}, \quad \dots, \quad V_{i_s}^{j_s} V_{i_s}^{k_s}$$

of s ($s \geq 1$) edges such that the copies

$$T_{i_1}, T_{i_2}, \dots, T_{i_s}$$

lie on the same side of the line segment

$$\vec{l} = V_{i_1}^{j_1} V_{i_s}^{k_s}.$$

To be more precise, the point $V_{i_r}^{k_r}$ coincides with the point $V_{i_{r+1}}^{j_{r+1}}$ for each $1 \leq r \leq s-1$. In this case, we write

$$\vec{l} = V_{i_1}^{j_1} V_{i_1}^{k_1} + V_{i_2}^{j_2} V_{i_2}^{k_2} + \dots + V_{i_s}^{j_s} V_{i_s}^{k_s}$$

Assume that there also holds

$$\vec{l} = V_{i'_1}^{j'_1} V_{i'_1}^{k'_1} + V_{i'_2}^{j'_2} V_{i'_2}^{k'_2} + \dots + V_{i'_t}^{j'_t} V_{i'_t}^{k'_t}$$

such that all the copies

$$T_{i'_1}, T_{i'_2}, \dots, T_{i'_t}$$

lie on the other side of \vec{l} , as illustrated in Figure 2.

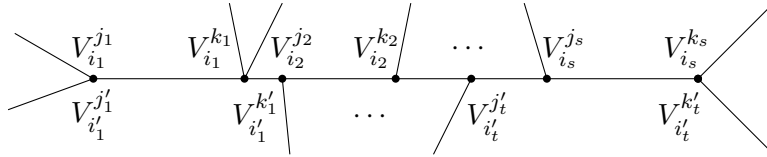


Figure 2. The arrangement (2.3).

In this case, we say that \vec{l} is represented in \mathcal{T} , denoted as

$$S\left(V_{i_1}^{j_1} V_{i_1}^{k_1}, \quad V_{i_2}^{j_2} V_{i_2}^{k_2}, \quad \dots, \quad V_{i_s}^{j_s} V_{i_s}^{k_s}\right) = \left[V_{i'_1}^{j'_1} V_{i'_1}^{k'_1}, \quad V_{i'_2}^{j'_2} V_{i'_2}^{k'_2}, \quad \dots, \quad V_{i'_t}^{j'_t} V_{i'_t}^{k'_t}\right]. \quad (2.3)$$

Denote by \mathbb{Z} the set of integers, by \mathbb{Z}^+ the set of positive integers, and by \mathbb{N} the set of nonnegative integers. Let $N > 0$ and $a_1, \dots, a_5 \in \mathbb{N}$. We say that the set

$$\{a_1 V^1, a_2 V^2, \dots, a_5 V^5\} \quad (2.4)$$

is a sum- N -collection of the angles if

$$N = \sum_{i=1}^5 a_i V^i.$$

We call (2.4) a sum- N -collection to the angle V^j if $a_j \geq 1$. For convenience, we remove the term $a_i V^i$ from the collection (2.4) if

$$a_i = 0.$$

Denote by $R_N(V^j; T)$ the set of sum- N -collections to V^j in T . For simplifying notation, we write

$$R_{2\pi}(V^j; T) = R(V^j; T).$$

Since T is a tiler, we have

$$R(V^j; T) \neq \emptyset$$

for any $1 \leq j \leq 5$. For example, if

$$V^1 = V^2 = \frac{5\pi}{6},$$

$$V^3 = \frac{2\pi}{3},$$

$$V^4 = V^5 = \frac{\pi}{3},$$

then we have

$$R(V^1; T) = \{\{2V^1, V^4\}, \{2V^1, V^5\}, \{V^1, V^2, V^4\}, \{V^1, V^2, V^5\}\},$$

$$R_\pi(V^3; T) = \{\{V^3, V^4\}, \{V^3, V^5\}\}.$$

Note that the sum of any four angles of a pentagon is larger than 2π , while the sum of any three angles is larger than π . This leads to the following lemma immediately.

Lemma 2.1 *Any sum- 2π -collection (2.4) has at most three positive a_j , while any sum- π -collection (2.4) has at most two positive a_j .*

3 The slight-rotating operation

In this section, we first demonstrate the selection of the initial cross-section subject to some technical conditions. With the initial cross-section in hand, we can then introduce the slight-rotating operation which will play the essential role in determining all pentahedron universal tilers.

Denote by \mathcal{E} the set of pentahedron universal tilers without parallel facets. The goal of this paper is to prove that

$$\mathcal{E} = \emptyset.$$

We will do this by contradiction. It is well-known that pentahedra have two distinct topological types. One is the quadrilateral-based pyramids, the other is composed of a pair of triangular bases and three quadrilateral sides, such as triangular prisms. In particular, we see that any pentahedron has a quadrilateral facet. Throughout this paper, we suppose that $\mathcal{P} \in \mathcal{E}$, and

$$Q = ABCD$$

is a convex quadrilateral facet of \mathcal{P} .

Let E be a point lying in the interior of the line segment AB such that E is neither A nor B . In this case, we express

$$E \in AB.$$

Set another point $F \in BC$.

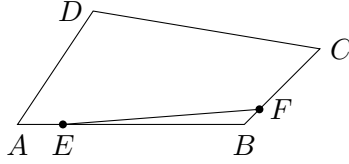


Figure 3. The pentagon $AEFCD$.

As illustrated in Figure 3, we have a convex pentagon

$$T = AEFCD.$$

The method of finding a cross-section by choosing $E \rightarrow A$ and $F \rightarrow B$ was used in [6], however, at this time, we need to select the points E and F more carefully. Before formulating the conditions for selecting E and F , we give the following lemma which will be frequently used in the proof of Lemma 3.2.

Lemma 3.1 *Let $N \geq 0$. Then both the set*

$$\{\theta > B/2 : N = a \cdot \theta + b \cdot \phi + c \cdot \psi, \ a \in \mathbb{Z}^+, \ b, c \in \mathbb{N}, \ \phi, \psi \in \{A, C, D\}\}$$

and the set

$$\{l < N : l = a \cdot CD + b \cdot DA + c \cdot AE, \ a, b \in \mathbb{N}, \ c \in \{0, \pm 1, \pm 2\}\}$$

have finite cardinalities.

It is easy to prove the above lemma and we omit the proof.

Lemma 3.2 *There exists $\delta_E > 0$ such that for any point $E \in AB$ with $AE < \delta_E$, there exists $\delta_F = \delta_F(E) > 0$ such that for any point $F \in BC$ with $BF < \delta_F$, the pentagon*

$$T = AEFCD$$

satisfies

(i) $2AE < \min\{EF, FC, CD, DA\}$;

(ii) $R(E; T) = R(F; T) \subseteq \{\{E, F, c\gamma\} : c \in \mathbb{Z}^+, \gamma \in \{A, C, D\}\}$.

(iii) for any $a, b \in \mathbb{N}$, $c \in \{0, \pm 1, \pm 2\}$, and $l \in \{EF, 2FC\}$, there holds

$$l \neq a \cdot CD + b \cdot DA + c \cdot AE.$$

Proof. Choose a point $E \in AB$ and move it towards A such that the point E can be arbitrarily closed to but never arrive at the point A . Similarly, choose $F \in BC$ and move it towards B such that F can be arbitrarily closed to but never arrive at the point B . Let

$$x = \max\{AE, BF\}.$$

In the above moving procedure, it is clear that

$$\lim_{x \rightarrow 0} AE = 0, \quad (3.1)$$

$$\lim_{x \rightarrow 0} EF = AB, \quad (3.2)$$

$$\lim_{x \rightarrow 0} FC = BC, \quad (3.3)$$

$$\lim_{x \rightarrow 0} \angle AEF = \pi, \quad (3.4)$$

$$\lim_{x \rightarrow 0} \angle EFC = B. \quad (3.5)$$

By (3.1), (3.2) and (3.3), there exists δ_1 such that the condition (i) holds for any $x < \delta_1$.

By Lemma 2.1, we have

$$R(E; T) \subseteq \{\{aE, b\beta, c\gamma\} : a \in \mathbb{Z}^+, b, c \in \mathbb{N}, \beta, \gamma \in \{F, A, C, D\}\}. \quad (3.6)$$

By (3.4) and (3.5), there exists $\delta_2 < \delta_1$ such that for any $x < \delta_2$, there holds

$$2E + \alpha > 2\pi, \quad \forall \alpha \in \{A, F, C, D\}.$$

Let $x < \delta_2$. Then $a = 1$ in (3.6). By Lemma 3.1 and the limit (3.4), there exists $\delta_3 < \delta_2$ such that for any $x < \delta_3$, we have

$$E + b\beta + c\gamma \neq 2\pi, \quad \forall b, c \in \mathbb{N}, \beta, \gamma \in \{A, C, D\}.$$

Let $x < \delta_3$. In view of (3.6), we have

$$R(E; T) \subseteq \{\{E, bF, c\gamma\} : b \in \mathbb{Z}^+, c \in \mathbb{N}, \gamma \in \{A, C, D\}\}. \quad (3.7)$$

Note that

$$E + F = \pi + B. \quad (3.8)$$

So every sum- 2π -collection $\{E, bF, c\gamma\}$ with $b \geq 1$ corresponds to a sum- $(\pi - B)$ -collection

$$\{b'F, c\gamma\}, \quad (3.9)$$

where $b', c \in \mathbb{N}$ and $\gamma \in \{A, C, D\}$. By Lemma 3.1 and the limit (3.5), there exists $\delta_4 < \delta_3$ such that for any $x < \delta_4$, we have

$$b'F + c\gamma \neq \pi - B, \quad \forall b' \in \mathbb{Z}^+, c \in \mathbb{N}, \gamma \in \{A, C, D\}.$$

Let $x < \delta_4$. In view of (3.9), we find $b' = 0$ and thus $b = 1$. By (3.8), we see that $c \geq 1$ in (3.7). This proves that

$$R(E; T) \subseteq \{ \{E, F, c\gamma\} : c \in \mathbb{Z}^+, \gamma \in \{A, C, D\} \}. \quad (3.10)$$

In the same vein, we have

$$R(F; T) \subseteq \{ \{aF, b\beta, c\gamma\} : a \in \mathbb{Z}^+, b, c \in \mathbb{N}, \beta, \gamma \in \{E, A, C, D\} \}. \quad (3.11)$$

By Lemma 3.1 and the limit (3.5), there exists $\delta_5 < \delta_4$ such that for any $x < \delta_5$,

$$aF + b\beta + c\gamma \neq 2\pi, \quad \forall a \in \mathbb{Z}^+, b, c \in \mathbb{N}, \beta, \gamma \in \{A, C, D\}.$$

In view of (3.11), we deduce that

$$R(F; T) \subseteq \{ \{aF, bE, c\gamma\} : a, b \in \mathbb{Z}^+, c \in \mathbb{N}, \gamma \in \{A, C, D\} \}.$$

So any sum- 2π -collection to F is a sum- 2π -collection to E . By (3.10), we obtain that

$$R(F; T) = R(E; T).$$

This proves the condition (ii).

Let $\delta_E = \delta_5$, and let $E \in AB$ with $AE < \delta_E$. By Lemma 3.1, there exists δ_F depending on the choice of E such that the condition (iii) holds for any $F \in BC$ with $BF < \delta_F$. This completes the proof. \blacksquare

We remark that the points E and F can be chosen from any other pair of adjacent edges of Q , subject to analogous conditions. This idea will be employed in the proof of Theorem 4.1.

We say that a cross-section is *proper* if none of its vertices is a vertex of \mathcal{P} . Let $\mathcal{C}_{\mathcal{P}}$ be the set of proper pentagonal cross-sections of \mathcal{P} . Fix two points

$$\begin{aligned} E^0 &\in AB, \\ F^0 &\in BC, \end{aligned}$$

satisfying the conditions (i)–(iii), and write the pentagon

$$P^0 = A^0 E^0 F^0 C^0 D^0, \quad (3.12)$$

where $A^0 = A$, $C^0 = C$, and $D^0 = D$. Now we recursively define a sequence $\{P^k\}_{k \geq 1}$ of proper pentagonal cross-sections.

Let π be a plane which crosses \mathcal{P} nontrivially. Let l be a line in π . For any $\varepsilon > 0$, denote by π_+^ε (resp. π_-^ε) the plane obtained by rotating π around l by the angle ε (resp. $-\varepsilon$). It is

clear that there exists ε such that at least one of the planes π_+^ε and π_-^ε crosses \mathcal{P} nontrivially. Write

$$p(\pi; l; \varepsilon) = \begin{cases} \pi_+^\varepsilon, & \text{if } \pi_+^\varepsilon \text{ crosses } \mathcal{P} \text{ nontrivially;} \\ \pi_-^\varepsilon, & \text{otherwise.} \end{cases}$$

Intuitively, the plane $p(\pi; l; \varepsilon)$ is obtained by rotating π a little along l . For simplifying notation, we use $\text{cr}(\pi; l; \varepsilon)$ to denote the intersection $C(p(\pi; l; \varepsilon))$ of the plane $p(\pi; l; \varepsilon)$ and the polyhedron \mathcal{P} .

Since every vertex of Q has valence 3, there exists $\delta_0 > 0$ such that

$$\text{cr}(P^0; E^0 F^0; \varepsilon) \in \mathcal{C}_{\mathcal{P}}, \quad \forall 0 < \varepsilon \leq \delta_0.$$

Define

$$P^1 = \text{cr}(P^0; E^0 F^0; \delta_0) = A^1 E^1 F^1 C^1 D^1 \quad (3.13)$$

to be the initial cross-section. In particular, we have

$$\begin{aligned} E^1 &= E^0, \\ F^1 &= F^0. \end{aligned}$$

Suppose that $P^k \in \mathcal{C}_{\mathcal{P}}$ is well-defined for some $k \geq 1$. Let $e_1^k, e_2^k, \dots, e_5^k$ be the edges of P^k . It is clear that there exists $0 < \delta_k < \delta_{k-1}$ such that for any $0 < \varepsilon \leq \delta_k$ and any edge e_j^k of P^k , we have

$$\text{cr}(P^k; e_j^k; \varepsilon) \in \mathcal{C}_{\mathcal{P}}.$$

Choosing an edge e_j^k , we can define

$$P^{k+1} = \text{cr}(P^k; e_j^k; \delta_k). \quad (3.14)$$

Note that the cross-section P^{k+1} depends on the choices of e_j^k and δ_k ; while the value of δ_k depends on P^k but is independent of the choice of e_j^k . We call the above procedure of getting P^{k+1} from P^k the *slight-rotating operation*.

Since all cross-sections P^k are proper, the slight-rotating operation has a certain sign-preserving property if we take δ_k small enough. Denote by $\text{sgn}(x)$ the signum function, i.e., for any real number x ,

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$$

Let $i \geq 0$, $N > 0$, $1 \leq j \leq 5$ and $a_1, \dots, a_5, b_1, \dots, b_5 \in \mathbb{N}$. Let V_i^1, \dots, V_i^5 be the angles of P^i . Let

$$\begin{aligned} x_i &= \text{sgn}\left(N - \sum_{l=1}^5 a_l V_i^l\right), \\ y_i &= \text{sgn}\left(e_j^i - \sum_{l=1}^5 b_l e_l^i\right). \end{aligned}$$

If $x_i y_i \neq 0$, then there exists $0 < \delta \leq \delta_i$ such that for any $1 \leq j \leq 5$, the cross-section

$$P^{i+1} = \text{cr}(P^i; e_j^i; \delta)$$

satisfies

$$x_{i+1} = x_i,$$

$$y_{i+1} = y_i.$$

It is easy to show the above property if one regards P^{i+1} as a continuous function of the variable δ_i in the definitions (3.13) and (3.14). With the aid of this property, we deduce that each cross-section in the sequence $\{P^k\}_{k \geq 1}$ satisfies some conditions analogous to (i)—(iii) as if δ_i 's are small enough. For all $k \geq 2$, we name the vertices of the pentagon P^k by

$$P^k = A^k E^k F^k C^k D^k$$

in the natural way that $E^k \in AB$ and $F^k \in BC$.

Theorem 3.3 *For any $k \geq 1$, there exists $\delta_{k-1}^* \leq \delta_{k-1}$ such that for any edge $e_{j_1}^1$ of P^1 , any edge $e_{j_2}^2$ of P^2 , ..., and any edge $e_{j_{k-1}}^{k-1}$ of P^{k-1} , the cross-section P^k defined by (3.14) satisfies*

$$(i') \quad 2A^k E^k < \min\{E^k F^k, F^k C^k, C^k D^k, D^k A^k\};$$

$$(ii') \quad R(E^k; P^k) = R(F^k; P^k) \subseteq \{\{E^k, F^k, c\gamma\} : c \in \mathbb{Z}^+, \gamma \in \{A^k, C^k, D^k\}\};$$

$$(iii') \quad \text{for any } a, b \in \mathbb{N}, c \in \{0, \pm 1, \pm 2\} \text{ and } l \in \{E^k F^k, 2F^k C^k\}, \text{ there holds}$$

$$l \neq a \cdot C^k D^k + b \cdot D^k A^k + c \cdot A^k E^k;$$

$$(iv') \quad \text{if there exist } a \in \mathbb{Z}^+, b, c \in \mathbb{N}, N \in \{\pi, 2\pi\}, \text{ and three pairwise distinct angles } V_k^{j_1}, V_k^{j_2}, V_k^{j_3} \text{ of } P^k \text{ such that}$$

$$N = aV_k^{j_1} + bV_k^{j_2} + cV_k^{j_3},$$

then for any $0 \leq h \leq k$, the corresponding angles $V_h^{j_1}, V_h^{j_2}, V_h^{j_3}$ of P^h satisfy

$$N = aV_h^{j_1} + bV_h^{j_2} + cV_h^{j_3}.$$

Let δ_i^* ($i \geq 0$) be defined as in the above theorem.

Lemma 3.4 *Let $\mathcal{P} \in \mathcal{E}$ and $k \in \mathbb{Z}^+$. Suppose that*

$$\{E^k, F^k, A^k\} \in R(E^k; P^k). \quad (3.15)$$

Then we have

$$\{E^{k+1}, F^{k+1}, A^{k+1}\} \notin R(E^{k+1}; \text{cr}(P^k; C^k D^k; \delta_k^*)). \quad (3.16)$$

Similarly, if

$$\{E^k, F^k, C^k\} \in R(E^k; P^k),$$

then we have

$$\{E^{k+1}, F^{k+1}, C^{k+1}\} \notin R(E^{k+1}; \text{cr}(P^k; D^k A^k; \delta_k^*)). \quad (3.17)$$

Proof. By (3.15), we have

$$F^k C^k \parallel D^k A^k. \quad (3.18)$$

Write

$$\text{cr}(P^k; C^k D^k; \delta_k^*) = A^{k+1} E^{k+1} F^{k+1} C^k D^k.$$

If the statement (3.16) is false, then we have

$$F^{k+1} C^k \parallel D^k A^{k+1}. \quad (3.19)$$

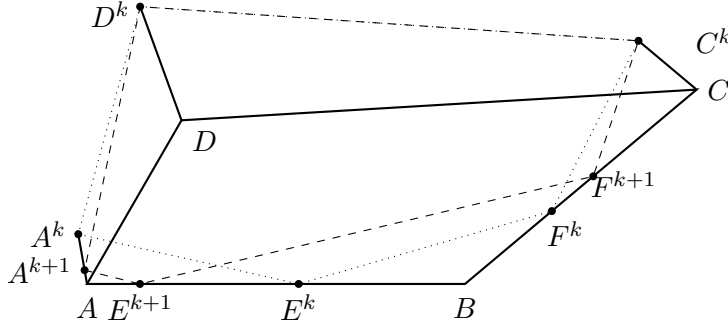


Figure 4. The parallel relation $F^k F^{k+1} C^k \parallel D^k A^k A^{k+1}$.

It is clear that the points F^k and F^{k+1} are distinct, while the points A^k and A^{k+1} are also distinct. Hence by (3.18) and (3.19), we find parallel facets

$$F^k F^{k+1} C^k \parallel D^k A^k A^{k+1},$$

see Figure 4. But $\mathcal{P} \in \mathcal{E}$, a contradiction. The relation (3.17) can be proved similarly. This completes the proof. \blacksquare

4 The main result

In this section, we determine all universal tiler by confirming that \mathcal{E} is empty.

Let $k \in \mathbb{Z}^+$. Suppose that \mathcal{T}_k is a tessellation of the plane by copies of P^k . Denote the copies used in \mathcal{T}_k by

$$\{P_i^k = A_i^k E_i^k F_i^k C_i^k D_i^k : i \in \Lambda_k\},$$

where Λ_k is a set. Note that each copy in \mathcal{T}_k are arranged counter-clockwise either in the order

$$A_i^k, E_i^k, F_i^k, C_i^k, D_i^k, \quad (4.1)$$

or in the order

$$A_j^k, D_j^k, C_j^k, F_j^k, E_j^k.$$

Denote by \mathcal{I}_k the set of indices $i \in \Lambda_k$ such that the vertices of P_i^k are arranged counter-clockwise in the order (4.1). Without loss of generality, we can invariably suppose that $1 \in \mathcal{I}_k$.

A quadrilateral is said to be cyclic if all its vertices lie on the same circle.

Theorem 4.1 *The facet Q is either a parallelogram or a cyclic quadrilateral.*

Proof. Suppose to the contrary that Q is neither a parallelogram nor cyclic. Without loss of generality, we can suppose that C is a largest angle of Q . Consider the cross-section

$$P^1 = \text{cr}(P^0; E^0 F^0; \delta_1^*).$$

For convenience, we rewrite the copies of P^1 as

$$P_i^1 = A_i E_i F_i C_i D_i.$$

Since P^1 is a tiler, we can express

$$S(E_1) = [E_1, \beta_j, X],$$

where

$$\beta_j \in \{A_j, E_j, F_j, C_j, D_j\},$$

and X is a sequence of angles. By the condition (ii'), we have

$$\beta_j \neq E_j.$$

Assume that

$$\beta_j = A_j.$$

By (i'), the sequence X contains no angle A_i . By (ii'), we deduce that

$$S(E_1) = [E_1, A_j, F_k] \tag{4.2}$$

for some $k \in \Lambda_1$, see Figure 5. It follows that

$$E^1 + A^1 + F^1 = 2\pi.$$

By the condition (iv'), we obtain

$$C + D = \pi.$$

So Q is a parallelogram, a contradiction.



Figure 5. The arrangements (4.2) and (4.3).

Below we can suppose that

$$\beta_j \in \{F_j, C_j, D_j\}.$$

In this case, the condition (i') implies

$$S(A_1) = [A_1, Y, \pi], \tag{4.3}$$

where Y is a sequence of angles, see Figure 5. By (i') and (ii') , we deduce that Y contains no angle A_i . Thus there exist $b \in \mathbb{Z}^+$ and $\beta^1 \in \{C^1, D^1\}$ such that

$$A^1 + b\beta^1 = \pi.$$

By (iv') , we find that

$$A + b\beta = \pi$$

for some $\beta \in \{C, D\}$. Since Q is neither a parallelogram nor cyclic, we see that $b \geq 2$. But C is a largest angle, so $\beta = D$. Namely,

$$A + bD = \pi. \tag{4.4}$$

Consider $E' \in AD$ and $F' \in CD$ subject to certain conditions corresponding to (i') — (iv') . Since C is a largest angle, we derive that

$$A + b'B = \pi \tag{4.5}$$

for some $b' \geq 2$. Adding (4.4) and (4.5) yields

$$2\pi = 2A + bD + b'B \geq 2(A + B + D) = 2(2\pi - C),$$

namely $C \geq \pi$, a contradiction. This completes the proof. ■

Theorem 4.2 *The facet Q is cyclic.*

Proof. Suppose to the contrary that Q is a non-cyclic facet. By Theorem 4.1, it is a parallelogram. Without loss of generality, we can suppose that C is a smallest angle of Q , and that

$$A + D = \pi.$$

Since Q is non-cyclic, we deduce that the angles C and D have distinct sizes. Therefore, the condition (ii) implies that

$$R(E^0; P^0) \subseteq \{\{E^0, F^0, A^0\}, \{E^0, F^0, C^0\}\}.$$

Consider the cross-sections (see Figure 6)

$$P^1 = \text{cr}(P^0; E^0 F^0; \delta_2^*) = A^1 E^1 F^1 C^1 D^1, \tag{4.6}$$

$$P^2 = \text{cr}(P^1; C^1 D^1; \delta_2^*) = A^2 E^2 F^2 C^2 D^2, \tag{4.7}$$

$$P^3 = \text{cr}(P^2; D^2 A^2; \delta_2^*) = A^3 E^3 F^3 C^3 D^3. \tag{4.8}$$

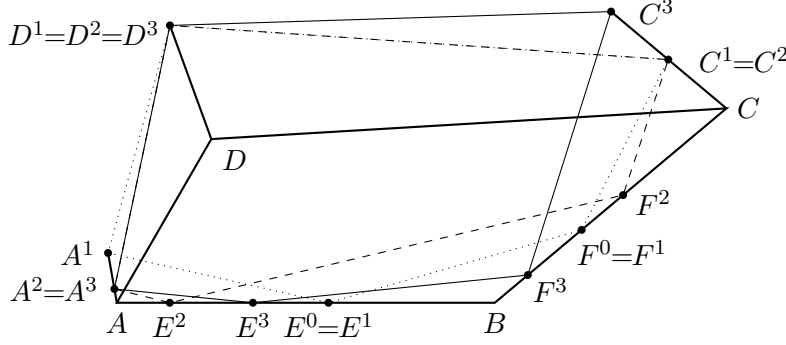


Figure 6. The cross-sections P^1 (dotted), P^2 (dashed) and P^3 (thin).

By Lemma 3.4 and (iv'), we see that

$$R(E^3; P^3) = \emptyset.$$

But the cross-section P^3 tiles the plane, a contradiction. This completes the proof. \blacksquare

To proceed further, we need the following technical lemma.

Lemma 4.3 *Let $k \geq 1$. Suppose that*

$$R(E^k; P^k) = R(F^k; P^k) = \{\{E^k, F^k, D^k\}\}. \quad (4.9)$$

Then any edge $F_i^k C_i^k$ is not represented. Moreover, any line segment

$$F_i^k C_i^k + C_j^k F_j^k$$

or

$$E_i^k A_i^k + C_j^k F_j^k$$

(if exists) is not represented.

Proof. For convenience, rewrite the copies $\{P_i^k : i \in \Lambda_k\}$ as

$$P_i^k = A_i E_i F_i C_i D_i.$$

Suppose to the contrary that $F_i C_i$ is represented. By (4.9), there is no point F_j ($j \neq i$) lying on the edge $F_i C_i$, and there is at most one point E_j lying on $F_i C_i$. Therefore, we have

$$F^k C^k = a_1 \cdot C^k D^k + b_1 \cdot D^k A^k + c_1 \cdot A^k E^k$$

for some $a_1, b_1 \in \mathbb{N}$ and $c_1 \in \{0, 1\}$. This contradicts to the condition (iii'). Hence $F_i C_i$ is not represented. For the same reason, any line segment

$$F_i C_i + C_j F_j$$

(if exists) is not represented.

Suppose that $1 \in \mathcal{I}_k$ and the line segment

$$E_1F_2 = E_1A_1 + C_2F_2$$

is represented in \mathcal{T}_k . Then $2 \in \mathcal{I}_k$, see Figure 7. We claim that there exist $i, j \in \Lambda_k$ such that

$$S(E_1) = [E_1, F_i, D_j]. \quad (4.10)$$

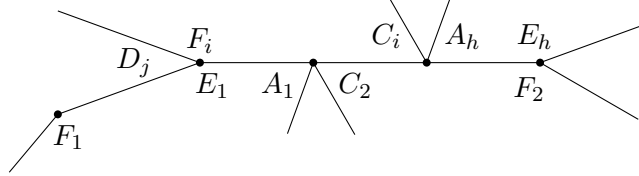


Figure 7. The tessellation of the ε -neighborhood of the line segment E_1F_2 .

In fact, by (4.9), there is at most one copy P_i^k such that the vertex E_i lying on the line segment E_1F_2 , where $i \neq 1, 2$. Also there is at most one point F_j ($j \neq 1, 2$) lying on the line segment E_1F_2 . If the sequence $S(E_1F_2)$ contains an edge E_iF_i , then the condition (ii') implies (4.10). Otherwise, by (iii'), we can deduce that $S(E_1F_2)$ contains an edge F_iC_i . Thus the condition (i') yields

$$S(E_1A_1, C_2F_2) = [F_iC_i, A_hE_h]$$

for some i and h . In particular, the relation (4.10) holds. This proves the claim.

By (4.10), the edge E_1F_1 is represented. It follows that there is at most one point E_i ($i \neq 1$) lying on the edge E_1F_1 , and there is no point F_j ($j \neq 1$) lying on E_1F_1 . So there exist $a_2, b_2 \in \mathbb{N}$ and $c_2 \in \{0, 1\}$ such that

$$E^kF^k = a_2 \cdot C^kD^k + b_2 \cdot D^kA^k + c_2 \cdot A^kE^k,$$

contradicting to the condition (iii'). This completes the proof. ■

Here is the main result of this paper.

Theorem 4.4 *A convex polyhedron is a universal tiler if and only if it is a tetrahedron or a triangular prism.*

Proof. It suffices to show that $\mathcal{E} = \emptyset$. Suppose to the contrary that $\mathcal{P} \in \mathcal{E}$. By Theorem 4.2, we see that Q is cyclic. Suppose that D is a smallest angle of Q . By the condition (ii), we have

$$R(E^0; P^0) \subseteq \left\{ \{E^0, F^0, A^0\}, \{E^0, F^0, C^0\}, \{E^0, F^0, D^0\} \right\}. \quad (4.11)$$

Consider the cross-sections P^1, P^2, P^3 defined by (4.6)–(4.8). By (4.11), Lemma 3.4 and the condition (iv'), we see that

$$R(E^3; P^3) \subseteq \{ \{E^3, F^3, D^3\} \}.$$

Since P^3 is a tiler, by (ii'), we have

$$R(E^3; P^3) = R(F^3; P^3) = \{\{E^3, F^3, D^3\}\}. \quad (4.12)$$

By (4.12) and (iv'), we deduce that

$$R_\pi(A^3; P^3) \subseteq \{\{A^3, C^3\}\} \cup \{\{bA^3, aD^3\} : b \in \mathbb{Z}^+, a \in \mathbb{N}\}. \quad (4.13)$$

Consider the tessellations of the plane by copies of P^3 . For convenience, rewrite

$$\begin{aligned} \mathcal{T} &= \mathcal{T}_3, \\ P_i^3 &= A_i E_i F_i C_i D_i, \\ \Lambda_3 &= \Lambda, \\ \mathcal{I}_3 &= \mathcal{I}. \end{aligned}$$

By (4.12) and (i'), there exist $m \in \mathbb{Z}^+$ and $i_1, i_2, \dots, i_m \in \Lambda_3$ such that

$$S(A_1) = [A_1, \alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_m}, \pi], \quad (4.14)$$

where

$$\alpha_{i_j} \in \{A_{i_j}, C_{i_j}, D_{i_j}\}.$$

Let $\varepsilon > 0$. Considering the tessellation of ε -neighborhood of the point E_{i_j} , we see that

$$\alpha_{i_j} \neq A_{i_j}, \quad \forall 1 \leq j \leq m. \quad (4.15)$$

Write $i_1 = 2$.

Assume that $m = 1$. If $\alpha_2 = D_2$, then

$$A^3 + D^3 = \pi$$

and thus

$$\{E^3, F^3, C^3\} \in R(E^3; P^3),$$

contradicting to (4.12). Therefore, by (4.15), the expression (4.14) reduces to

$$S(A_1) = [A_1, C_2, \pi],$$

see Figure 8. In this case, if $2 \notin \mathcal{I}$, then the point F_2 lies on the line determined by $D_1 A_1$. By (4.12), we find the edge $F_2 C_2$ represented, contradicting to Lemma 4.3. So $2 \in \mathcal{I}$. By (4.12), the line segment

$$E_1 A_1 + C_2 F_2$$

is represented, also contradicting to Lemma 4.3.

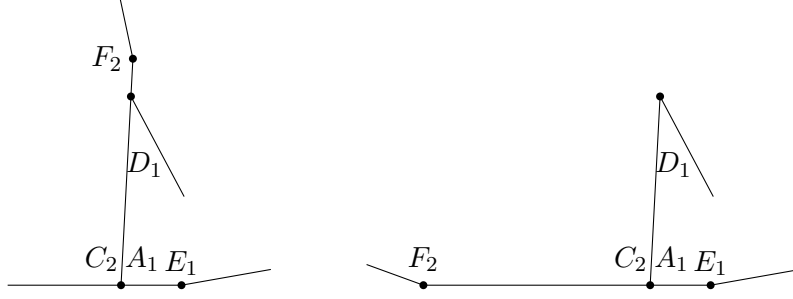


Figure 8. The tessellation for the case $m = 1$.

Below we can suppose that $m \geq 2$. Write $i_2 = 3$. By (4.14) and (4.15), the expression (4.14) reduces to

$$S(A_1) = [A_1, D_2, D_3, \dots, \pi]. \quad (4.16)$$

We have two cases depending on whether $2 \in \mathcal{I}$.

Assume that $2 \in \mathcal{I}$. By Lemma 4.3, the edge F_2C_2 is not represented. So there exist $s \in \mathbb{Z}^+$ and $j_1, j_2, \dots, j_s \in \Lambda_3$ such that

$$S(C_2) = [C_2, \pi, \beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_s}], \quad (4.17)$$

where β_i is an angle of the copy T_i . Therefore the edge D_2C_2 is represented. In view of (4.12), (4.16), and (4.17), there is no point E_i lying on the line segment D_2C_2 , neither is F_i . Moreover, by (4.12), no point A_j lies in the interior of D_2C_2 , since otherwise the ε -neighborhood of the point E_j can not be tiled. For the same reason, no point C_j lies in the interior of D_2C_2 . Therefore,

$$S(D_2C_2) \in \{[D_3A_3], [D_3C_3]\}.$$

So $j_s = 3$. If

$$S(D_2C_2) = [D_3A_3],$$

then either the edge A_3E_3 is represented (when $s \geq 2$), which is impossible by the condition (i') and (4.12); or the line segment

$$E_3A_3 + C_2F_2$$

is represented (when $s = 1$), which contradicts to Lemma 4.3. See Figure 9.

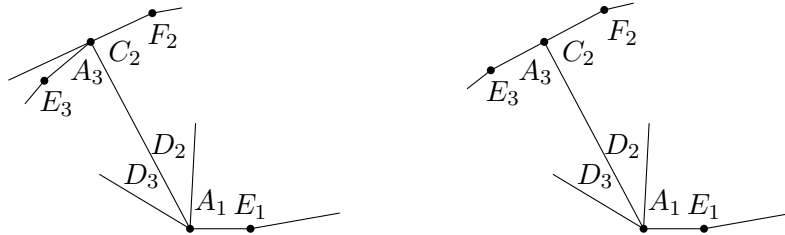


Figure 9. The tessellation for the case $S(D_2C_2) = [D_3A_3]$.

So

$$S(D_2C_2) = [D_3C_3].$$

In this case (see Figure 10), either F_3C_3 is represented (when $s \geq 2$), or

$$F_2C_2 + C_3F_3$$

is represented (when $s = 1$). Both of them contradicts to Lemma 4.3.

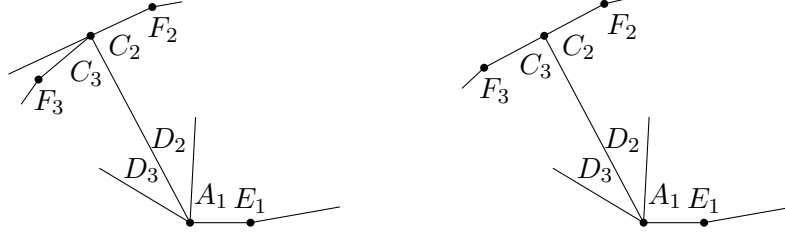


Figure 10. The tessellation for the case $S(D_2C_2) = [D_3C_3]$.

Hence we have $2 \notin \mathcal{I}$. Considering the tessellation of the ε -neighborhood of the point E_2 , we find that

$$S(A_2) = [A_2, \pi, \gamma_{h_1}, \gamma_{h_2}, \dots, \gamma_{h_t}], \quad (4.18)$$

where $t \geq 1$ and γ_{h_i} is an angle of the copy T_{h_i} . Therefore D_2A_2 is represented. For the same reason as in the case $2 \in \mathcal{I}$, we deduce that

$$S(D_2A_2) \in \{[D_3A_3], [D_3C_3]\}.$$

If

$$S(D_2A_2) = [D_3A_3],$$

then either A_3E_3 is represented (when $t \geq 2$) or

$$E_2A_2 + A_3E_3$$

is represented (when $t = 1$). See Figure 11. Both of them are absurd by the condition (i').

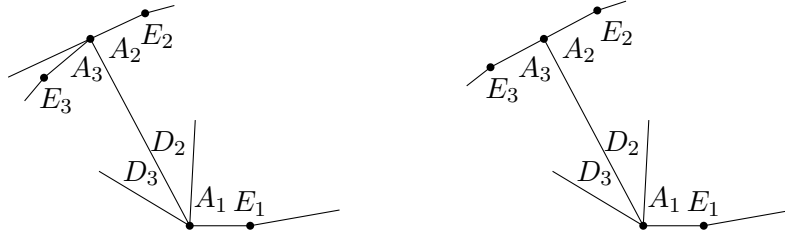


Figure 11. The tessellation for the case $S(D_2A_2) = [D_3A_3]$.

So

$$S(D_2A_2) = [D_3C_3].$$

In this case, either C_3F_3 is represented (when $t \geq 2$) or

$$F_3C_3 + A_2E_2$$

is represented. See Figure 12. Both of them contradicts to Lemma 4.3.

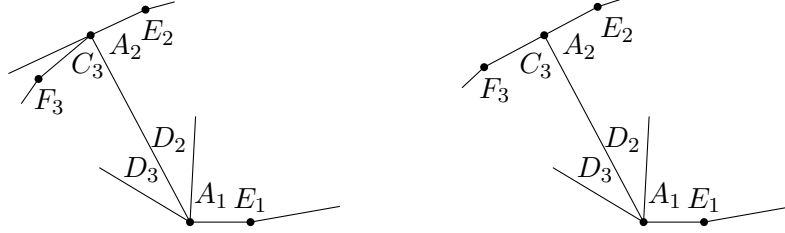


Figure 12. The tessellation for the case $S(D_2A_2) = [D_3C_3]$.

To conclude, the cross-section P^3 does not tile the plane. This implies that

$$\mathcal{E} = \emptyset,$$

and completes the proof. ■

References

- [1] J. Akiyama, Tile-makers and semi-tile-makers, Amer. Math. Monthly 114 (2007), 602–609.
- [2] B. Grünbaum and G. C. Shephard, Tilings and Patterns, W. H. Freeman and Company, New York, 1987.
- [3] M. D. Hirschhorn and D. C. Hunt, Equilateral convex pentagons which tile the plane, J. Combin. Theory Ser. A 39 (1985), 1–18.
- [4] K. Reinhardt, Über die Zerlegung der Ebene in Polygone, Dissertation, Universität Frankfurt, 1918.
- [5] T. Sugimoto and T. Ogawa, Properties of tilings by convex pentagons, Forma 21 (2006), 113–128.
- [6] D. G. L. Wang, On universal tilers, arXiv: math.CO/1109.0813.